INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 5.5]

OPTIMIZATION

Compiled by Christos Nikolaidis

A. Paper 1 questions (SHORT)

1. [Maximum mark: 6]

The perimeter of a rectangle is 24 metres.

(a) The table shows some of the possible dimensions of the rectangle.

Find the values of a, b, c, d and e.

Length (m)	Width (m)	Area (m^2)
1	11	11
a	10	b
3	c	27
4	d	e

[2]

(b) If the length of the rectangle is x m, and the area is A m^2, express A in terms of x only. [1]

(c) What are the length and width of the rectangle if the area is to be a maximum? [3]

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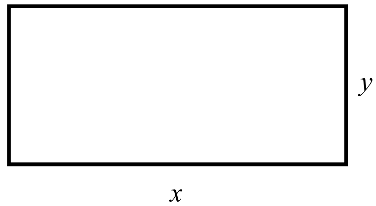
2. [Maximum mark: 8]

A rectangle has sides x and y and its perimeter is 40m.



(a) Express y in term of x. [1]

(b) Hence, express the area A of the rectangle in terms of x. [1]

(c) Use $\frac{\mathrm{d}A}{\mathrm{d}x}$ to find the maximum area of the rectangle; justify your answer. [5]

(d) What is the minimum value of A? [1]

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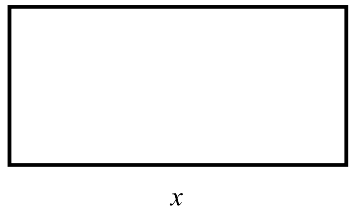
3. [Maximum mark: 9]

The area of a rectangle of length x is 100 m^2.



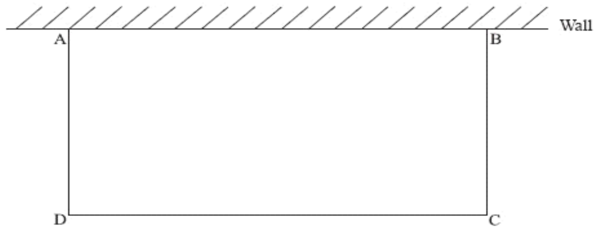
(a) Express the perimeter P of the rectangle in terms of x. [3]

(b) Use $\frac{\mathrm{d}P}{\mathrm{d}x}$ to find the minimum perimeter of the rectangle; justify your answer. [5]

(c) Explain why there is no maximum value for P. [1]

4. [Maximum mark: 7

The following diagram shows a rectangular area ABCD enclosed on three sides by 60 m of fencing, and on the fourth by a wall AB.



(a) Find the width of the rectangle that gives its maximum area. [6]

(b) Find the maximum area. [1]

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5. [Maximum mark: 7]

A farmer wishes to create a rectangular enclosure, ABCD, of area 525 m^2, as shown below.

The fencing used for side AB costs \$11 per metre. The fencing for the other three sides costs \$3 per metre. The farmer creates an enclosure so that the cost is a minimum. Find this minimum cost.

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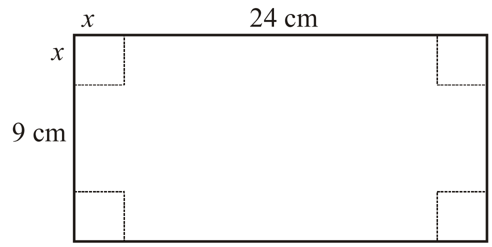
6. [Maximum mark: 8]

A rectangular piece of card measures 24 cm by 9 cm. Equal squares of length x cm are cut from each corner of the card as shown in the diagram below. What is left is then folded to make an **open** box, of length l cm and width w cm.



(a) Write expressions, in terms of x, for

(i) the length l. (ii) the width w. [2]

(b) Show that the volume (B m^3) of the box is given by $B = 4x^3 - 66x^2 + 216x$. [1]

(c) Find $\frac{dB}{dx}$. [1]

(d) (i) Find the value of x which gives the maximum volume of the box.

(ii) Calculate the maximum volume of the box. [4]

7. [Maximum mark: 10]

The cost of producing a mathematics textbook is $ 15 and it is then sold for $ x.

(a) Find an expression for the profit made on each book sold. [1]

A total of (100 000 – 4000 x) books is sold.

(b) Show that the profit made on all the books sold is $P = 160\,000x - 4000x^2 - 1500\,000$ [3]

(c) (i) Find $\frac{dP}{dx}$.

(ii) Hence calculate the value of x to make a maximum profit [4]

(d) Calculate the number of books sold to make this maximum profit. [2]

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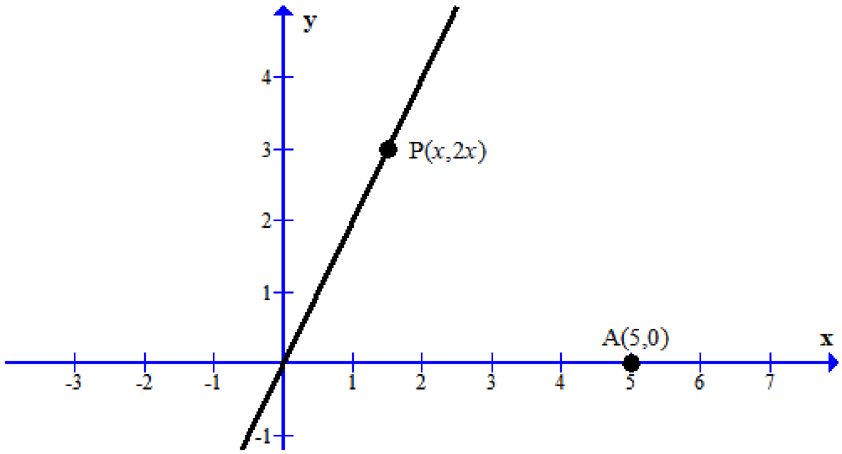
8. [Maximum mark: 7]

A point on the line $y = 2x$ has coordinates $P(x, 2x)$. Let A(5,0).



(a) Show that the distance D between P and A is given by $D = \sqrt{5x^2 - 10x + 25}$ [1]

Let $S = D^2 = 5x^2 - 10x + 25$

(b) Find $\frac{dS}{dx}$. [2]

(c) **Hence,** find

(i) the minimum value of S

(ii) the minimum distance D between the line and the point A

(iii) the coordinates of the point P on the line which is closest to A. [4]

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9. [Maximum mark: 9]

A point on the curve $y = x^2$ has coordinates $\text{P}\left(a, a^2\right)$. The point $\text{A}(3,0)$ lies on x-axis.

(a) Show that the distance D between P and A is given by $D = \sqrt{a^4 + a^2 - 6a + 9}$ [1]

Let $S = D^2 = a^4 + a^2 - 6a + 9$

(b) (i) Find $\frac{\mathrm{d}S}{\mathrm{d}a}$

(ii) Show that $a = 1$ is a stationary point for S. Justify that it gives a minimum. [5]

(c) Hence or otherwise find

(i) the coordinates of the point P on the curve which is closest to A.

(ii) the minimum distance D between the curve and the point A. [3]

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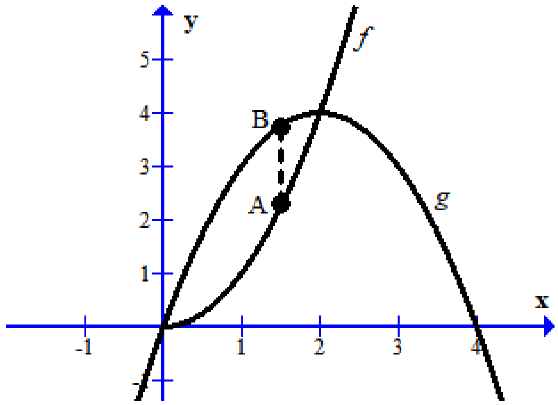
10. [Maximum mark: 9]

The following diagram shows parts of the graphs of $f(x)=x^2$ and $g(x)=4x-x^2$.



The two graphs intersect at $x=0$ and $x=a$.

(a) Find the value of a. [3]

A vertical line segment [AB] is drawn between the curves in the interval $0 \le x \le a$ where A is on $y=f(x)$ and B is on $y=g(x)$, as shown above.

(b) Express the length L of the line segment [AB] in terms of x. [1]

(c) Find the maximum value of L, justifying it is a maximum. [5]

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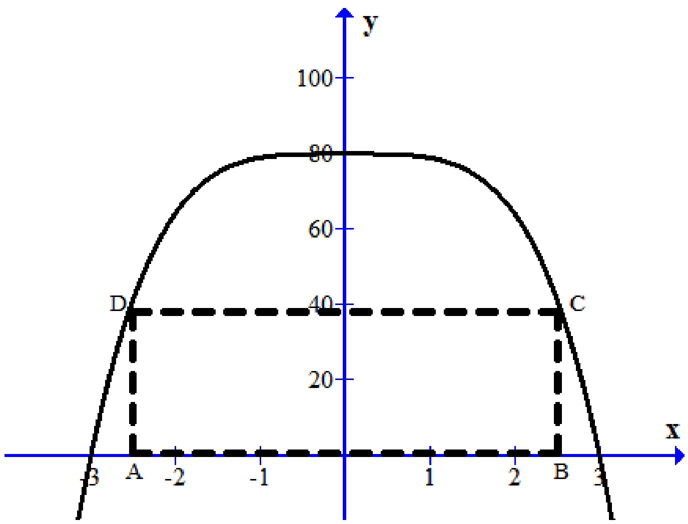
11. [Maximum mark: 7]

Let $f(x)=80-x^4$. A rectangle ABCD is inscribed in the region between the curve $y=f(x)$ and x-axis as shown in the diagram below. Assume that he lower vertices of the rectangle have coordinates A($-a$,0) and B(a,0)



(a) Show that the area of the rectangle is given by $S=160a-2a^5$. [2]

(b) Find the maximum value S, justifying it is a maximum. [5]

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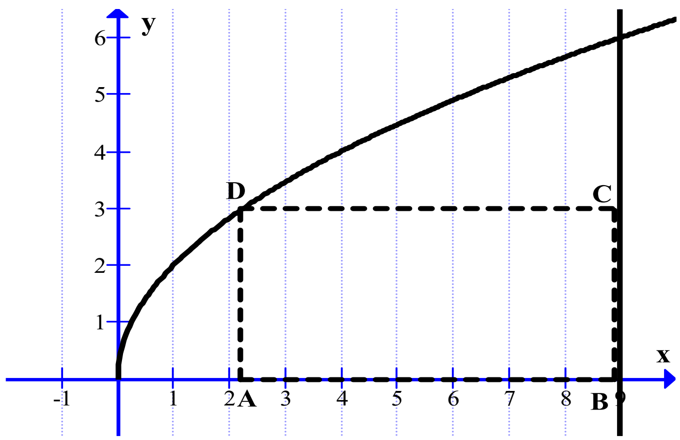
12. [Maximum mark: 9]

A rectangle ABCD is enclosed between the x-axis, the curve $y = 2\sqrt{x}$, and the vertical line $x = 9$, so that the lower vertices are on the x-axis with coordinates A(a,0) and B(9,0) respectively, the vertex D is on the curve, while the vertex C is on the vertical line. The area of this rectangle is denoted by *S*.



(a) Find an expression of *S* in terms of a. [2]

(b) Find the maximum value of *S*; justify your answer. [5]

(c) Write down the values of a for which we obtain a rectangle of minimum area *S*. [2]

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B. Paper 2 questions (LONG)

13. [Maximum mark: 13]

The height (cm) of a daffodil above the ground is given by $h(w) = 24w - 2.4w^2$ where w is the time in weeks after the plant has broken through the surface ($w \geq 0$).

(a) Calculate the height of the daffodil after two weeks. [2]

(b) (i) Find the rate of growth, $\frac{\mathrm{d}h}{\mathrm{d}w}$.

(ii) The rate of growth when $w = k$ is 7.2 cm per week. Find k .

(iii) When will the daffodil reach its maximum height? What height will it reach? [8]

(c) Once the daffodil has reached its maximum height, it begins to fall back towards the ground. Show that it will touch the ground after 70 days. [3]

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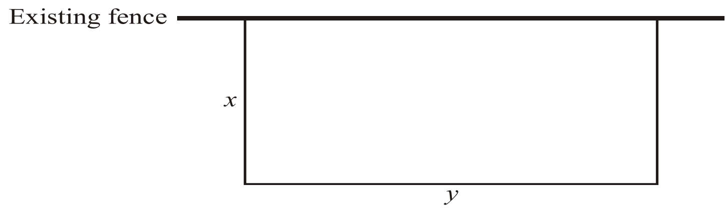
14. [Maximum mark: 11]

A farmer wishes to enclose a rectangular field using an existing fence for one of the four sides.



(a) Write an expression in terms of x and y for the total length of the new fence.

(b) The farmer has enough materials for 2500 metres of new fence. Show that

$$y = 2500 - 2x$$ [1]

(c) $A(x)$ represents the area of the field in terms of x.

(i) Show that $A(x) = 2500x - 2x^2$.

(ii) Find $A'(x)$.

(iii) Hence or otherwise find the value of x that produces the max area of the field.

(iv) Find the maximum area of the field. [8]

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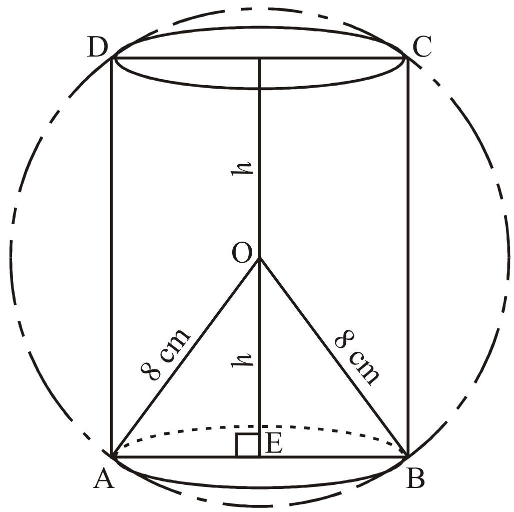
15. [Maximum mark: 12]

A cylinder is cut from a solid wooden sphere of radius 8 cm as shown in the diagram.

The height of the cylinder is $2h$ cm.



(a) Find AE (the radius of the cylinder), in terms of h. [2]

(b) Show that the volume (V) of the cylinder may be written as $V = 2\pi h(64 - h^2)\,\text{cm}^3$. [2]

(c) (i) Determine, correct to three significant figures, the height of the cylinder with the greatest volume that can be produced in this way.

(ii) Calculate this greatest volume; give your answer correct to the nearest cm^3. [8]

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16. [Maximum mark: 13]

A closed box has a square base of side x and height h.

(a) Write down an expression for the volume, V, of the box. [1]

(b) Write down an expression for the total surface area, A, of the box. [1]

The volume of the box is 1000 cm^3

(c) Express h in terms of x. [2]

(d) Hence show that $A = 4000x^{-1} + 2x^2$. [2]

(e) Find $\frac{dA}{dx}$. [2]

(f) Calculate the value of x that gives a minimum surface area. [3]

(g) Find the surface area for this value of x. [2]

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